

# Solving Dirichlet's Problem Constructively

**Maarten McKubre-Jordens**

**"To find oneself lost in wonder at some manifestation  
is frequently the half of a new discovery."**

**– J.P.G. Lejeune Dirichlet**

Throughout, we work in the Bishop-style constructive mathematical framework (**BISH**), which is mathematics using intuitionistic logic and some appropriate constructive foundation (e.g. Aczel's CZF or Martin-Löf type theory).

Occasionally we may refer to classical mathematics (**CLASS**) for comparison purposes.

# BHK Interpretation

Intuitionistic logic is positively characterized by the Brouwer-Heyting-Kolmogorov interpretation of the logical connectives. We emphasize:

$P \Rightarrow Q$  We have an algorithm which turns proofs of  $P$  into proofs of  $Q$ .

$\exists_{x \in A} P(x)$  We have an algorithm which  
(i) computes an object,  $x$ , belonging to  $A$ , and  
(ii) shows that  $P(x)$ .

# Observations:

Some points to note about mathematics using intuitionistic logic:

- Classical theorems often become more enlightening when given constructive proof.
- Often there are multiple classically identical theorems that are constructively distinguishable.

## Observations:

Some points to note about mathematics using intuitionistic logic:

- Classical theorems often become more enlightening when given constructive proof.
- Often there are multiple classically identical theorems that are constructively distinguishable.
- Constructive proofs embody (in principle) an algorithm for computing objects.
- A constructive proof also proves that the algorithm it embodies is correct (i.e. meets its design specification).

Here we present conditions which ensure that if we can construct weak solutions to the Dirichlet problem on arbitrary close internal approximations to a domain  $\Omega \subset \mathbb{R}^n$  then we can construct a (unique) weak solution on  $\Omega$  also.

We give some Brouwerian examples to show that, even on geometrically quite reasonable domains in  $\mathbb{R}^2$ , existence of (weak or strong) solutions is an *essentially nonconstructive* result.

# The Dirichlet Problem

asks us to identify a function  $u$  (with some suitable regularity conditions) such that

$$\Delta u = f \text{ on } \Omega, \quad u(x) = 0 \text{ for all } x \in \partial\Omega,$$

where  $\Omega$  is a domain in  $\mathbb{R}^n$  and  $f \in L_2(\Omega)$ .

Alternatively we may consider

$$\Delta u = 0 \text{ on } \Omega, \quad u(x) = f(x) \text{ for all } x \in \partial\Omega$$

for some uniformly continuous  $f$ .

By a *weak solution* we mean an element  $u \in H_0^1(\Omega)$  such that

$$\langle u, v \rangle_{H_0^1(\Omega)} \equiv - \int_{\Omega} \nabla u \cdot \nabla v = \int_{\Omega} f v$$

for all  $v \in C_0^1(\Omega)$ .

# The Dirichlet problem

Associated is the minimisation problem for the Dirichlet energy functional:

*Find a twice-differentiable continuous function  $u$  which minimises*

$$J(u) = \int_{\Omega} (\|\nabla u\|^2 + 2uf) \, dx.$$

(the classical proof of equivalence is essentially constructive)



# Why are classical approaches not constructive?

Classical approach for finding a weak solution:

- Define a linear functional on  $H_0^1(\Omega)$  by

$$\varphi(v) = - \int_{\Omega} v f \, dx.$$

This is a bounded functional.

# Why are classical approaches not constructive?

Classical approach for finding a weak solution:

- Define a linear functional on  $H_0^1(\Omega)$  by

$$\varphi(v) = - \int_{\Omega} v f \, dx.$$

This is a bounded functional.

- Apply the Riesz representation theorem to get  $u \in H_0^1(\Omega)$  with

$$\varphi(v) = \langle u, v \rangle$$

for all  $v \in H_0^1(\Omega)$ ; then  $u$  is what we want.

# Why are classical approaches not constructive?

Classical approach for finding a weak solution:

- Define a linear functional on  $H_0^1(\Omega)$  by

$$\varphi(v) = - \int_{\Omega} v f \, dx.$$

This is a bounded functional.

- Apply the Riesz representation theorem to get  $u \in H_0^1(\Omega)$  with

$$\varphi(v) = \langle u, v \rangle$$

for all  $v \in H_0^1(\Omega)$ ; then  $u$  is what we want.

This is not constructive!

# Why are classical approaches not constructive?

Classical approach to minimising  $J(u)$ :

- Compute the infimum of  $J(u)$ , guaranteed to exist since  $J(u)$  is bounded below.

It can be shown that  $J(u) \geq -2\gamma^2 \int_{\Omega} |f|^2 dx$ .

# Why are classical approaches not constructive?

Classical approach to minimising  $J(u)$ :

- Compute the infimum of  $J(u)$ , guaranteed to exist since  $J(u)$  is bounded below.

It can be shown that  $J(u) \geq -2\gamma^2 \int_{\Omega} |f|^2 dx$ .

- Construct a minimising sequence  $(u_n)_{n \geq 1}$ .

$(u_n)_{n \geq 1}$  is uniformly bounded in  $H_0^1(\Omega)$ .

# Why are classical approaches not constructive?

Classical approach to minimising  $J(u)$ :

- Compute the infimum of  $J(u)$ , guaranteed to exist since  $J(u)$  is bounded below.

It can be shown that  $J(u) \geq -2\gamma^2 \int_{\Omega} |f|^2 dx$ .

- Construct a minimising sequence  $(u_n)_{n \geq 1}$ .

$(u_n)_{n \geq 1}$  is uniformly bounded in  $H_0^1(\Omega)$ .

- Use weak sequential compactness of bounded sets in  $H_0^1(\Omega)$ .

We get weak convergence to the solution  $u$ .

# Why are classical approaches not constructive?

Classical approach to minimising  $J(u)$ :

- Compute the infimum of  $J(u)$ , guaranteed to exist since  $J(u)$  is bounded below.

It can be shown that  $J(u) \geq -2\gamma^2 \int_{\Omega} |f|^2 dx$ .

- Construct a minimising sequence  $(u_n)_{n \geq 1}$ .  
 $(u_n)_{n \geq 1}$  is uniformly bounded in  $H_0^1(\Omega)$ .

- Use weak sequential compactness of bounded sets in  $H_0^1(\Omega)$ .

We get weak convergence to the solution  $u$ .

This is also not constructive!

Constructive  
Mathematics

Dirichlet's  
Problem

Measure &  
Geometry

Sufficient  
conditions

Nonconstructive  
Principles

Weak  
Solvability  
implies LPO

Solvability  
implies WLPO

Summary

It turns out a whole lot depends on the domain. The condition that  $\Omega$  is open, bounded, and integrable is *not constructively sufficient*.



A subset  $S$  of a metric space  $X$  is *well contained* in  $\Omega \subset X$  if there exists  $r > 0$  such that  $x \in \Omega$  whenever  $\rho(x, S) < r$ .

A subset  $\Omega$  of a metric space  $X$  is:

- *located* if  $\rho(x, \Omega) = \inf\{\rho(x, y) : y \in \Omega\}$  exists (i.e. can be computed) for each  $x \in X$ .
- *totally bounded* if for each  $\varepsilon > 0$  it has a finite  $\varepsilon$ -approximation.
- *edge coherent* if  $x \in \Omega$  whenever  $x \in \overline{\Omega}$  and  $\rho(x, \partial\Omega) > 0$ ,

Further, by a *complemented set*  $\Omega$  in  $X$  we mean an ordered pair  $(\Omega^1, \Omega^0) \subset X \times X$  such that  $\rho(x, y) > 0$  for all  $x \in \Omega^1, y \in \Omega^0$ .

If the characteristic function of a (complemented) set  $\Omega$  is integrable then we call  $\Omega$  an integrable set with measure  $\mu(\Omega) = \int \chi_\Omega$ .

We define a subset  $\Omega$  of  $\mathbb{R}^n$  a *Wang domain* if:

- It is edge coherent, totally bounded and open in  $\mathbb{R}^n$ .
- The boundary  $\partial\Omega$  is compact.
- The (complemented) set  $\Omega \equiv (\Omega, -\Omega)$  is Lebesgue integrable and, together with  $\partial\Omega$ , subject to the divergence theorem.
- There exists  $c_0 > 0$  such that if

$$(\partial\Omega)_r = \{x \in \overline{\Omega} : \rho(x, \partial\Omega) \leq r\}$$

is integrable, then

$$\int_{(\partial\Omega)_r} |u|^2 \leq c_0 r^2 \int_{(\partial\Omega)_r} \|\nabla u\|^2 \quad (u \in H_0^1(\Omega))$$

(a Poincaré-like inequality—this gives control over  $u$  near the boundary).

## Theorem

*Let  $\Omega$  be a Wang domain in  $\mathbb{R}^n$  and let  $f \in L_2(\Omega)$ . Suppose there exists a sequence  $(\Omega_n)_{n \geq 1}$  of edge coherent, totally bounded, open subsets of  $\Omega$ , each having compact boundary, such that for each  $n$ ,*

- $\overline{\Omega_n} \subset\subset \Omega_{n+1}$ ,
- $\Omega_n$  and  $\partial\Omega_n$  are Lebesgue integrable, with  $\mu(\partial\Omega_n) = 0$ ,
- $\max \{ \rho(\overline{\Omega_n}, \overline{\Omega}), \rho(\partial\Omega_n, \partial\Omega) \} \rightarrow 0$  as  $n \rightarrow \infty$ , and
- *the Dirichlet problem has a weak solution  $u_n$  in  $H_0^1(\Omega_n)$ .*

*Then the sequence  $(u_n)_{n \geq 1}$  converges in  $H_0^1(\Omega)$  to a weak solution of the Dirichlet problem.*

## Outline of proof

- Compute  $\kappa > 0$  such that if  $\varepsilon > 0$  and  $K \subset \mathbb{R}^n$  is compact there is a  $C^\infty$  cutoff function  $\eta : \mathbb{R}^n \rightarrow [0, 1]$ :

$$\|\nabla \eta(x)\| \leq \frac{\kappa}{\varepsilon} \quad \text{almost a characteristic function but nicer.}$$

## Outline of proof

- Compute  $\kappa > 0$  such that if  $\varepsilon > 0$  and  $K \subset \mathbb{R}^n$  is compact there is a  $C^\infty$  cutoff function  $\eta : \mathbb{R}^n \rightarrow [0, 1]$ :

$$\|\nabla \eta(x)\| \leq \frac{\kappa}{\varepsilon} \quad \text{almost a characteristic function but nicer.}$$

- Approximate  $\Omega_n$  with well-contained compact subset  $K_n$ .
- Estimate  $\int \|\nabla u_n\|^2$  (and hence  $\int |u_n|^2$ ) on  $\Omega_n - K$  appropriately, and show that  $(u_n)_{n \geq 1}$  is Cauchy.

## Outline of proof

- Compute  $\kappa > 0$  such that if  $\varepsilon > 0$  and  $K \subset \mathbb{R}^n$  is compact there is a  $C^\infty$  cutoff function  $\eta : \mathbb{R}^n \rightarrow [0, 1]$ :

$$\|\nabla \eta(x)\| \leq \frac{\kappa}{\varepsilon} \quad \text{almost a characteristic function but nicer.}$$

- Approximate  $\Omega_n$  with well-contained compact subset  $K_n$ .
- Estimate  $\int \|\nabla u_n\|^2$  (and hence  $\int |u_n|^2$ ) on  $\Omega_n - K$  appropriately, and show that  $(u_n)_{n \geq 1}$  is Cauchy.
- Use completeness of  $H_0^1(\Omega)$  to compute its limit,  $u$ .

## Outline of proof

- Compute  $\kappa > 0$  such that if  $\varepsilon > 0$  and  $K \subset \mathbb{R}^n$  is compact there is a  $C^\infty$  cutoff function  $\eta : \mathbb{R}^n \rightarrow [0, 1]$ :

$$\|\nabla \eta(x)\| \leq \frac{\kappa}{\varepsilon} \quad \text{almost a characteristic function but nicer.}$$

- Approximate  $\Omega_n$  with well-contained compact subset  $K_n$ .
- Estimate  $\int \|\nabla u_n\|^2$  (and hence  $\int |u_n|^2$ ) on  $\Omega_n - K$  appropriately, and show that  $(u_n)_{n \geq 1}$  is Cauchy.
- Use completeness of  $H_0^1(\Omega)$  to compute its limit,  $u$ .

The function  $u$  solves the Dirichlet problem.

Constructive  
Mathematics

Dirichlet's  
Problem

Measure &  
Geometry

Sufficient  
conditions

Nonconstructive  
Principles

Weak  
Solvability  
implies LPO

Solvability  
implies WLPO

Summary

## Intermission : Dirichlet (1805–1859)



\* Last name: Lejeune Dirichlet.



Constructive  
Mathematics

Dirichlet's  
Problem

Measure &  
Geometry

Sufficient  
conditions

Nonconstructive  
Principles

Weak  
Solvability  
implies LPO

Solvability  
implies WLPO

Summary

## Intermission : Dirichlet (1805–1859)



\* Last name: Lejeune Dirichlet.

\* Awarded an honorary doctorate.  
(1927)

Constructive  
Mathematics

Dirichlet's  
Problem

Measure &  
Geometry

Sufficient  
conditions

Nonconstructive  
Principles

Weak  
Solvability  
implies LPO

Solvability  
implies WLPO

Summary

## Intermission : Dirichlet (1805–1859)



- \* Last name: Lejeune Dirichlet.
- \* Awarded an honorary doctorate.  
(1927)
- \* Supported 1848 revolution.  
(The revolution failed.)

Constructive  
Mathematics

Dirichlet's  
Problem

Measure &  
Geometry

Sufficient  
conditions

Nonconstructive  
Principles

Weak  
Solvability  
implies LPO

Solvability  
implies WLPO

Summary

## Intermission : Dirichlet (1805–1859)



- \* Last name: Lejeune Dirichlet.
- \* Awarded an honorary doctorate.  
(1927)
- \* Supported 1848 revolution.  
(The revolution failed.)
- \* Gauss's successor at Göttingen.

Constructive  
Mathematics

Dirichlet's  
Problem

Measure &  
Geometry

Sufficient  
conditions

Nonconstructive  
Principles

Weak  
Solvability  
implies LPO

Solvability  
implies WLPO

Summary

## Intermission : Dirichlet (1805–1859)



- \* Last name: Lejeune Dirichlet.
- \* Awarded an honorary doctorate.  
(1927)
- \* Supported 1848 revolution.  
(The revolution failed.)
- \* Gauss's successor at Göttingen.

His brain is preserved at the University of Göttingen.

Constructive  
Mathematics

Dirichlet's  
Problem

Measure &  
Geometry

Sufficient  
conditions

Nonconstructive  
Principles

Weak  
Solvability  
implies LPO

Solvability  
implies WLPO

Summary

## Intermission : Dirichlet (1805–1859)



- \* Last name: Lejeune Dirichlet.
- \* Awarded an honorary doctorate.  
(1927)
- \* Supported 1848 revolution.  
(The revolution failed.)
- \* Gauss's successor at Göttingen.

His brain is preserved at the University of Göttingen.

He is credited with in part establishing the modern definition of function.

## Summary

(Youngest sister of  
Felix Mendelssohn.)



Constructive  
Mathematics

Dirichlet's  
Problem

Measure &  
Geometry

Sufficient  
conditions

Nonconstructive  
Principles

Weak  
Solvability  
implies LPO

Solvability  
implies WLPO

Summary

His wife, Rebecka  
Mendelssohn.  
(1811–1858)

(Youngest sister of  
Felix Mendelssohn.)



“My older brother and sister stole my reputation as an artist. In any other family I would have been highly regarded as a musician and perhaps been leader of a group. Next to Felix and Fanny, I could not aspire to any recognition.”

## Some important principles

The *Limited Principle of Omniscience*: given any binary sequence  $(a_n)_{n \geq 1}$ , we can prove

$$\text{LPO: } \forall n(a_n = 0) \vee \exists n(a_n = 1).$$

The *Weak Limited Principle of Omniscience*: given any binary sequence  $(a_n)_{n \geq 1}$ , we can prove

$$\text{WLPO: } \forall n(a_n = 0) \vee \neg \forall n(a_n = 0).$$

*Markov's Principle* of unbounded search: given any binary sequence  $(a_n)_{n \geq 1}$ ,

$$\text{MP: } \neg \forall n(a_n = 0) \Rightarrow \exists n(a_n = 1).$$



# Construction 1

Let  $(a_n)_{n \geq 1}$  be an increasing binary sequence with  $a_1 = 0$ .

Let  $D$  be the open unit disc in  $\mathbb{R}^2$  and for each positive integer  $n$  set

$$T_n = \left\{ x \in \mathbb{R}^2 : \frac{1}{n} < \|x\| < 1 \right\}.$$

# Construction 1

Let  $(a_n)_{n \geq 1}$  be an increasing binary sequence with  $a_1 = 0$ .

Let  $D$  be the open unit disc in  $\mathbb{R}^2$  and for each positive integer  $n$  set

$$T_n = \left\{ x \in \mathbb{R}^2 : \frac{1}{n} < \|x\| < 1 \right\}.$$

- If  $a_n = 0$ , set  $\Omega_n = D$ .

# Construction 1

Let  $(a_n)_{n \geq 1}$  be an increasing binary sequence with  $a_1 = 0$ .

Let  $D$  be the open unit disc in  $\mathbb{R}^2$  and for each positive integer  $n$  set

$$T_n = \left\{ x \in \mathbb{R}^2 : \frac{1}{n} < \|x\| < 1 \right\}.$$

- If  $a_n = 0$ , set  $\Omega_n = D$ .
- If  $a_n = 1 - a_{n-1}$  set  $\Omega_k = T_n$  for all  $k \geq n$ .

# Construction 1

Let  $(a_n)_{n \geq 1}$  be an increasing binary sequence with  $a_1 = 0$ .

Let  $D$  be the open unit disc in  $\mathbb{R}^2$  and for each positive integer  $n$  set

$$T_n = \left\{ x \in \mathbb{R}^2 : \frac{1}{n} < \|x\| < 1 \right\}.$$

- If  $a_n = 0$ , set  $\Omega_n = D$ .
- If  $a_n = 1 - a_{n-1}$  set  $\Omega_k = T_n$  for all  $k \geq n$ .

Let  $\Omega$  be the interior of the subset  $\bigcap_{n \geq 1} \Omega_n$  of  $D$ .

Constructive  
Mathematics

Dirichlet's  
Problem

Measure &  
Geometry

Sufficient  
conditions

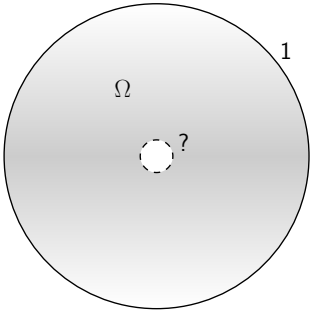
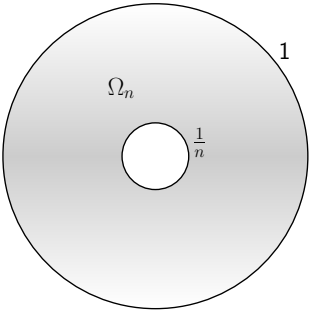
Nonconstructive  
Principles

Weak  
Solvability  
implies LPO

Solvability  
implies WLPO

Summary

# Construction 1



# Weak solutions imply **LPO**

Take  $\Omega$  from Construction 1, and let  $f(x) = \log \|x\|$ .

It can be shown that  $f$  is (uniformly!) continuous on  $\partial\Omega$ .  
(Subtlety: we need **MP** for this.)

## Weak solutions imply LPO

Take  $\Omega$  from Construction 1, and let  $f(x) = \log \|x\|$ .

It can be shown that  $f$  is (uniformly!) continuous on  $\partial\Omega$ .  
(Subtlety: we need **MP** for this.)

Suppose that the Dirichlet problem

$$\Delta u = 0 \text{ on } \Omega, \quad u(x) = f(x) \text{ for all } x \in \partial\Omega$$

has a weak solution in  $H_0^1(\Omega)$ .

If  $a_n = 0$  for all  $n$ , then  $u \equiv 0$  is the (weak and strong) solution of the Dirichlet problem on  $\Omega$ , and in this case

$$\int_{\Omega} u^2 = 0.$$

But if  $a_n = 1$  for some  $n$ , then  $u \equiv \log \|x\|$  is the (weak and strong) solution, and then:

$$\int_{\Omega} u^2 = \pi \left( \frac{1}{2} - \frac{1}{n^2} \log^2 \frac{1}{n} + \frac{1}{n^2} \log \frac{1}{n} - \frac{1}{2n^2} \right).$$

This is increasing for  $n > 0$ , its limit as  $n \rightarrow \infty$  is  $\frac{\pi}{2}$ , and the case  $n = 2$  is larger than  $\frac{1}{4}$ .



But if  $a_n = 1$  for some  $n$ , then  $u \equiv \log \|x\|$  is the (weak and strong) solution, and then:

$$\int_{\Omega} u^2 = \pi \left( \frac{1}{2} - \frac{1}{n^2} \log^2 \frac{1}{n} + \frac{1}{n^2} \log \frac{1}{n} - \frac{1}{2n^2} \right).$$

This is increasing for  $n > 0$ , its limit as  $n \rightarrow \infty$  is  $\frac{\pi}{2}$ , and the case  $n = 2$  is larger than  $\frac{1}{4}$ .

Since we can decide whether

$$\int_{\Omega} u^2 > 0 \quad \text{or} \quad \int_{\Omega} u^2 < \frac{1}{4}$$

we have **LPO**. (again using **MP**)

## Theorem

*The following are equivalent over **BISH**:*

- (i) **MP** holds, and for every totally bounded, Lebesgue integrable open subset of  $\mathbb{R}^n$  and every uniformly continuous function  $f : \partial\Omega \rightarrow \mathbb{R}$  the Dirichlet problem has a weak solution in  $H_0^1(\Omega)$ .
- (ii) **LPO**.

# Outline of proof ((ii) implies (i))

- Define a linear functional

$$\varphi(v) = - \int_{\Omega} v f$$

Standard estimates show  $|\varphi(v)| \leq \gamma \|f\|_2 \|v\|_{H_0^1(\Omega)}.$

## Outline of proof ((ii) implies (i))

- Define a linear functional

$$\varphi(v) = - \int_{\Omega} v f$$

Standard estimates show  $|\varphi(v)| \leq \gamma \|f\|_2 \|v\|_{H_0^1(\Omega)}$ .

- Construct an orthonormal basis  $\{e_1, e_2, \dots\}$  for  $H_0^1(\Omega)$ .  
For each  $n$ , consider the restriction to  $H_n$  of  $\varphi$ :

$$\|\varphi|_{H_n}\| = \sup\{|\varphi(v)| : v \in H_n\} \quad \text{exists.}$$

## Outline of proof ((ii) implies (i))

- Define a linear functional

$$\varphi(v) = - \int_{\Omega} v f$$

Standard estimates show  $|\varphi(v)| \leq \gamma \|f\|_2 \|v\|_{H_0^1(\Omega)}$ .

- Construct an orthonormal basis  $\{e_1, e_2, \dots\}$  for  $H_0^1(\Omega)$ .  
For each  $n$ , consider the restriction to  $H_n$  of  $\varphi$ :

$$\|\varphi|_{H_n}\| = \sup\{|\varphi(v)| : v \in H_n\} \quad \text{exists.}$$

- Use **LPO** to obtain the existence of  $\lim_{n \rightarrow \infty} \|\varphi|_{H_n}\|$ .  
(This gives normability of  $\varphi$ .)

## Outline of proof ((ii) implies (i))

- Define a linear functional

$$\varphi(v) = - \int_{\Omega} v f$$

Standard estimates show  $|\varphi(v)| \leq \gamma \|f\|_2 \|v\|_{H_0^1(\Omega)}$ .

- Construct an orthonormal basis  $\{e_1, e_2, \dots\}$  for  $H_0^1(\Omega)$ .  
For each  $n$ , consider the restriction to  $H_n$  of  $\varphi$ :

$$\|\varphi|_{H_n}\| = \sup\{|\varphi(v)| : v \in H_n\} \quad \text{exists.}$$

- Use **LPO** to obtain the existence of  $\lim_{n \rightarrow \infty} \|\varphi|_{H_n}\|$ .  
(This gives normability of  $\varphi$ .)

- Apply the Riesz representation theorem to get a function  $u$  that solves the Dirichlet problem.

## Observations about $\Omega$

- It satisfies an *exterior cone condition*, but not uniformly (**LPO**).
- It is edge coherent and (perhaps more importantly) Lipschitz.
- Approximating it internally by compact sets is nonconstructive (**WLPO**).
- The boundary  $\partial\Omega$  is not totally bounded (**WLPO**).
- Although  $\Omega$  is located,  $\partial\Omega$  is not.
- $(\partial\Omega)_r$  is not constructively integrable (**WLPO**).

## Observations about $\Omega$

- It satisfies an *exterior cone condition*, but not uniformly (**LPO**).
- It is edge coherent and (perhaps more importantly) Lipschitz.
- Approximating it internally by compact sets is nonconstructive (**WLPO**).
- The boundary  $\partial\Omega$  is not totally bounded (**WLPO**).
- Although  $\Omega$  is located,  $\partial\Omega$  is not.
- $(\partial\Omega)_r$  is not constructively integrable (**WLPO**).
- We used **MP**.



Constructive  
Mathematics

Dirichlet's  
Problem

Measure &  
Geometry

Sufficient  
conditions

Nonconstructive  
Principles

Weak  
Solvability  
implies LPO

Solvability  
implies WLPO

Summary

## Corollary

*If every open, Lebesgue integrable domain in  $\mathbb{R}^n$  has integrable boundary, then **WLPO** holds.*

So there are subtleties concerning the divergence theorem.

Constructive  
Mathematics

Dirichlet's  
Problem

Measure &  
Geometry

Sufficient  
conditions

Nonconstructive  
Principles

Weak  
Solvability  
implies LPO

Solvability  
implies WLPO

Summary

## Corollary

*If every open, Lebesgue integrable domain in  $\mathbb{R}^n$  has integrable boundary, then **WLPO** holds.*

So there are subtleties concerning the divergence theorem.

We may state a suitably modified version (omitted).

## Construction 2

Let  $(a_n)_{n \geq 1}$  be a binary sequence.

Let  $D$  be the open unit disc in  $\mathbb{R}^2$  and for each positive integer  $n$  set

$$P_n = D - \left\{ x \in \mathbb{R}^2 : \|x\| < 1 \wedge 0 \leq \arg x \leq \frac{1}{n} \right\}.$$

## Construction 2

Let  $(a_n)_{n \geq 1}$  be a binary sequence.

Let  $D$  be the open unit disc in  $\mathbb{R}^2$  and for each positive integer  $n$  set

$$P_n = D - \left\{ x \in \mathbb{R}^2 : \|x\| < 1 \wedge 0 \leq \arg x \leq \frac{1}{n} \right\}.$$

■ If  $a_n = 0$ , set  $\Omega_n = D$ .

## Construction 2

Let  $(a_n)_{n \geq 1}$  be a binary sequence.

Let  $D$  be the open unit disc in  $\mathbb{R}^2$  and for each positive integer  $n$  set

$$P_n = D - \left\{ x \in \mathbb{R}^2 : \|x\| < 1 \wedge 0 \leq \arg x \leq \frac{1}{n} \right\}.$$

- If  $a_n = 0$ , set  $\Omega_n = D$ .
- If  $a_n = 1$  set  $\Omega_n = P_n$ .

## Construction 2

Let  $(a_n)_{n \geq 1}$  be a binary sequence.

Let  $D$  be the open unit disc in  $\mathbb{R}^2$  and for each positive integer  $n$  set

$$P_n = D - \left\{ x \in \mathbb{R}^2 : \|x\| < 1 \wedge 0 \leq \arg x \leq \frac{1}{n} \right\}.$$

- If  $a_n = 0$ , set  $\Omega_n = D$ .
- If  $a_n = 1$  set  $\Omega_n = P_n$ .

Let  $\Omega$  be the interior of the subset  $\bigcap_{n \geq 1} \Omega_n$  of  $D$ .

Constructive  
Mathematics

Dirichlet's  
Problem

Measure &  
Geometry

Sufficient  
conditions

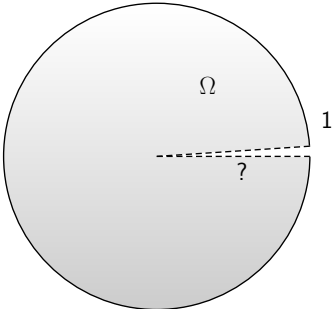
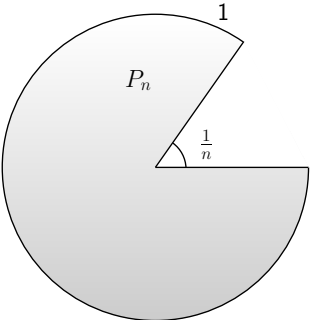
Nonconstructive  
Principles

Weak  
Solvability  
implies LPO

Solvability  
implies WLPO

Summary

# Construction 2



# Strong solutions imply **WLPO**

Take  $\Omega$  from Construction 2, and let  $f(x) = 1 - \|x\|$ .

Clearly  $f$  is uniformly continuous on  $\partial\Omega$ .



## Strong solutions imply **WLPO**

Take  $\Omega$  from Construction 2, and let  $f(x) = 1 - \|x\|$ .

Clearly  $f$  is uniformly continuous on  $\partial\Omega$ .

If  $a_n = 0$  for all  $n$ , then  $u \equiv 0$  is the (strong) solution of the Dirichlet problem and so  $u(0,0) = 0$ .

But if  $a_n = 1$  for some  $n$ , then  $(0,0) \in \partial\Omega$  and the boundary condition forces  $u(0,0) = 1$ .

## Strong solutions imply **WLPO**

Take  $\Omega$  from Construction 2, and let  $f(x) = 1 - \|x\|$ .

Clearly  $f$  is uniformly continuous on  $\partial\Omega$ .

If  $a_n = 0$  for all  $n$ , then  $u \equiv 0$  is the (strong) solution of the Dirichlet problem and so  $u(0,0) = 0$ .

But if  $a_n = 1$  for some  $n$ , then  $(0,0) \in \partial\Omega$  and the boundary condition forces  $u(0,0) = 1$ .

Thus if this Dirichlet problem has a strong solution, **WLPO** holds.

## Theorem

*If for every totally bounded, Lebesgue integrable open subset of  $\mathbb{R}^n$  and every (uniformly) continuous function  $f : \partial\Omega \rightarrow \mathbb{R}$  the Dirichlet problem*

$$\Delta u = 0 \text{ on } \Omega, \quad u(x) = f(x) \text{ for all } x \in \partial\Omega$$

*has a strong solution then **WLPO** holds.*

Note that the domain from Construction 2 satisfies the same properties as that of Construction 1.

## Theorem

*If for every totally bounded, Lebesgue integrable open subset of  $\mathbb{R}^n$  and every (uniformly) continuous function  $f : \partial\Omega \rightarrow \mathbb{R}$  the Dirichlet problem*

$$\Delta u = 0 \text{ on } \Omega, \quad u(x) = f(x) \text{ for all } x \in \partial\Omega$$

*has a strong solution then **WLPO** holds.*

Note that the domain from Construction 2 satisfies the same properties as that of Construction 1.

The converse is not known to hold.

# Lesson

The question of existence of (weak or strong) solutions to Dirichlet problems, as classically conceived, is *essentially nonconstructive*. In other words,

# Lesson

The question of existence of (weak or strong) solutions to Dirichlet problems, as classically conceived, is *essentially nonconstructive*. In other words,

*There is no universal algorithm for computing solutions to the classical Dirichlet problem.*

# Lesson

The question of existence of (weak or strong) solutions to Dirichlet problems, as classically conceived, is *essentially nonconstructive*. In other words,

*There is no universal algorithm for computing solutions to the classical Dirichlet problem.*

(Perhaps surprising, since a 1979 paper by Y.K. Chan constructs Green's functions for a wide class of domains.)

# Lesson

The question of existence of (weak or strong) solutions to Dirichlet problems, as classically conceived, is *essentially nonconstructive*. In other words,

*There is no universal algorithm for computing solutions to the classical Dirichlet problem.*

(Perhaps surprising, since a 1979 paper by Y.K. Chan constructs Green's functions for a wide class of domains.)

We have provided *sufficient conditions* for solutions to be constructible.

Question: What conditions are *necessary*?



Constructive  
Mathematics

Dirichlet's  
Problem

Measure &  
Geometry

Sufficient  
conditions

Nonconstructive  
Principles

Weak  
Solvability  
implies LPO

Solvability  
implies WLPO

Summary

## A corollary

The Dirichlet problem corresponds to a special stationary case of the Navier-Stokes equations of fluid flow. It is now immediate from the previous result that:

## A corollary

The Dirichlet problem corresponds to a special stationary case of the Navier-Stokes equations of fluid flow. It is now immediate from the previous result that:

*There is no universal algorithm for computing solutions to the classical Navier-Stokes equations.*

Constructive  
Mathematics

Dirichlet's  
Problem

Measure &  
Geometry

Sufficient  
conditions

Nonconstructive  
Principles

Weak  
Solvability  
implies LPO

Solvability  
implies WLPO

Summary

# Acknowledgements

- Douglas Bridges
- Michael Wang
- University of Canterbury

